ITC (Phnom – Penh)

Préparation de Mathématiques

(Concours FUI)

Octobre 2022

Module 9 Power series

Gilbert Monna

APESAM

Module 9 Power series

I) Radius of convergence

All the sequences considered in this module are real or complex, the symbol $|.|$ denoting the absolute value.

1) Definitions

We denote by $(a_n)_{n\in\mathbb{N}}$ a real or complex sequence.
We call associated power series the series of real or complex functions whose general term is defined by:

$$\forall n \in \mathbb{N},\ u_n(z) = a_n z^n,\ \forall z \in \mathbb{R} \text{ or } \mathbb{C}.$$

2) Fundamental theorem

Let $\sum\limits_{n\geq 0} a_n z^n$ be a power series. If there exists a complex number z such that the series is not absolutely convergent, there exists one and only one element R of $[0, +\infty[$:
If $|z| < R$, the series $\sum\limits_{n\geq 0} a_n z^n$ is absolutely convergent.
If $|z| > R$, the series $\sum\limits_{n\geq 0} a_n z^n$ is divergent and its general term does not go towards 0.
Moreover: $R = \sup\{r \in \mathbb{R}^+ \ / \ \sum\limits_{n\geq 0} |a_n| r^n \text{ converges}\}$.

3) Practical determination of the radius of convergence

a) Principle of majoring and minoring series
Let $\sum\limits_{n\geq 0} a_n z^n$ and $\sum\limits_{n\geq 0} b_n z^n$ be two power series of radius of convergence R_a and R_b.
We assume that for any natural integer n, $|a_n| \leqslant |b_n|$. Then $R_a \geqslant R_b$.

b) D'Alembert's rules
$\sum\limits_{n\geq 0} a_n z^n$ is a power series and we assume that there existe $N \in \mathbb{N}$ such that:
$\forall n \geqslant N,\ an \neq 0$.
If $\lim\limits_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = L$ strictly positive real number, the direct application of the D'Alembert's rule for the numeric positive series gives:
If $|z| < \frac{1}{L}$ the series $\sum\limits_{n\geq 0} a_n z^n$ is convergent.
If $|z| > \frac{1}{L}$ the series $\sum\limits_{n\geq 0} a_n z^n$ is divergent.
Therefore $R = \lim\limits_{n\to+\infty} \frac{|a_n|}{|a_{n+1}|}$.
If $L = 0$, the series is convergent for any complex number z and the radius of convergence is infinite.
There is of course no true inverse.

II) Fundamental properties

1) Theorem of normal convergence

A power series of radius R is normally convergent on any closed disc contained in the open disc of center 0 and of radius R.
Let us assume $r < R$ and $|z| \leqslant r$.
$|a_n z^n| \leqslant |a_n| r^n$, $\forall z \in \overline{D}(0,r)$ and the series of general term $|a_n| r^n$ is convergent by definition of the radius of convergence. This implies the normal convergence of the series on the closed disc of center 0 and radius r.

Remark
There is no general result on the open disc of radius R, nor on the closed disc: the series can have any behavior on the border of the convergence disc $D(0,R)$, some series are normaly convergent on the closed disc of convergence, other diverge in any point of the border of the disc...

2) Differentiation of a power series

Theorem
The complex power series $\sum_{n\geq 0} a_n z^n$ and $\sum_{n\geq 1} n a_n z^{n-1}$ have the same radius of convergence.

Theorem of differentiation
Let $\sum_{n\geq 0} a_n x^n$ be a real power series of radius R.

For any real number x such that $|x| < R$, we set $f(x) = \sum_{n=0}^{+\infty} a_n x^n$.
The function f is differentiable on $]-R,R[$ and $f'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$.

Theorem fundamental
Let $\sum_{n\geq 0} a_n x^n$ be a real power series of radius R, the function defined on interval $]-R,R[$ by $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ is of class $\mathcal{C}^\infty$, and for any integer p, the derivative of order p is:

$$f^{(p)}(x) = \sum_{n=p}^{+\infty} n(n-1)\dots(n-p+1) a_n x^{n-p}.$$

In particular : $\forall n \in \mathbb{N},\ a_n = \frac{f^{(n)}(0)}{n!}$.

3) Integration

As for differentiation, we only consider here real power series.
The theorem of integration term to term of normaly convergent series apply on segments included in the open interval of convergence (in $\mathbb{R}$ the discs are intervals).
If the radius of convergence of the series $\sum_{n\geq 0} a_n x^n$ is R, finite or infinite, for any

pair (a,b) of real numbers such that $[a,b]$ are included in $]-R,R[$, by setting: $f(x)=\sum\limits_{n=0}^{+\infty}a_nx^n$ for any x element of $]-R,R[$ we have:

$$\int\limits_a^b f(x)dx=\sum_{n=0}^{+\infty}a_n\int\limits_a^b x^n dx.$$

We use only this result to find a development in power series of an antiderivative of function f :

$$F(x)=\int\limits_0^x f(x)dx=\sum_{n=0}^{+\infty}a_n\int\limits_0^x t^n dt=\sum_{n=0}^{+\infty}\frac{a_nx^{n+1}}{n+1}.$$

This implies that the power series $\sum\limits_{n\geq 0}\frac{a_nx^n}{n+1}$ has a radius of convergence greater or equal to R.
We only need to apply the theorem of derivation to prove that it is indeed R.

4) Theorem of radius convergence of Abel-Dirichlet
Let $(a_n)_{n\in\mathbb{N}}$ be a complex sequence such that the real power series $\sum\limits_{n\geq 0}a_nx^n$ is of radius of convergence R.
If the series $\sum\limits_{n\geq 0}a_nR^n$ converges, then the function defined by $f(x)=\sum\limits_{n=0}^{+\infty}a_nx^n$ is continuous on $[0,R]$.
We have the same result on $[-R,0]$.

III) Sums and products of power series

1) Sum

Theorem
Let $\sum\limits_{n\geq 0}a_nz^n$ and $\sum\limits_{n\geq 0}b_nz^n$ be two power series of radiuses R_a and R_b. The power series $\sum\limits_{n\geq 0}(a_n+b_n)\,z^n$ has a radius of convergence greater or equal to $\inf(R_a,R_b)$, with equality in the case where the radiuses are distinct.

2) Product

a) Product of convolution
Defintion
Let $\sum\limits_{n\geq 0}a_nz^n$ and $\sum\limits_{n\geq 0}b_nz^n$ be two power series, we call product series of power series $\sum\limits_{n\geq 0}a_nz^n$ and $\sum\limits_{n\geq 0}b_nz^n$ the power series $\sum\limits_{n\geq 0}\sum\limits_{p=0}^{n}a_pb_{n-p}z^n$.

Theorem
Let $\sum\limits_{n\geq 0}a_nz^n$ and $\sum\limits_{n\geq 0}b_nz^n$ be two power series of radiuses R_a and R_b.
Then the power series $\sum\limits_{n\geq 0}\sum\limits_{p=0}^{n}a_pb_{n-p}z^n$ ha a radius of convergence greater or

equal to $\inf(R_a, R_b)$ and for any z such that $|z| < \inf(R_a, R_b)$:

$$\sum_{n=0}^{+\infty}\sum_{p=0}^{n} a_p b_{n-p} z^n = \left(\sum_{n=0}^{+\infty} a_n z^n\right)\left(\sum_{n=0}^{+\infty} b_n z^n\right).$$

IV) Development in power series of a function

1) Definition, unicity, existence

a) Function to develop in power series

Being given a function f of real or complex variable, we say that f can be developed in power series around 0 if there exists a power series $\sum\limits_{n\geq 0} a_n z^n$, of non zero radius R, and a disc D of center 0 and of radius r, $r \leqslant R$, such that:

$$\forall z \in D,\ f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

b) Unicity of the development in power series

If two power series $\sum\limits_{n\geq 0} a_n z^n$ and $\sum\limits_{n\geq 0} b_n z^n$ are such that there exists a disc $D(0,\rho)$, ρ being strictly positive and lower than both radiuses of convergence, such that $\sum\limits_{n=0}^{+\infty} a_n z^n = \sum\limits_{n=0}^{+\infty} b_n z^n$, for any z belonging to $D(0,\rho)$, then $a_n = b_n$ for any n.

c) Condition necessary for a function can be developed in power series

For a function f to be developed in a power series, it is necessary that f is a function of class $\mathcal{C}^\infty$ around 0, and we have:
$a_n = \frac{f^{(n)}(0)}{n!}$ if $\sum\limits_{n\geq 0} a_n z^n$ is the development of f in power series.

Remarks

To any function f of class $\mathcal{C}^\infty$ around 0, we can associate the sequence $a_n = \frac{f^{(n)}(0)}{n!}$ then the power series $\sum\limits_{n\geq 0} a_n z^n$, called Taylor-series of f.
The unicity of the development in power series and the previous result imply that if the development in power series of a function exists, it is equal to its Taylor-series.
The condition f of class $\mathcal{C}^\infty$ is not enough, here is one of the most famous conter-examples:
$f(x) = e^{-\frac{1}{x^2}}$ if x is different of 0, $f(0) = 0$.
$\lim\limits_{x\to 0} f(x) = 0$, by compared growths, hence f is continuous in 0. We know by the general theorems that f is of class $\mathcal{C}^\infty$ on $\mathbb{R}^*$.
$f'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}}$, hence f' admits a zero limit when x goes towards 0, which shows that f is differentiable in 0 of zero limit.
We then show by recurrence that $f^{(n)}(x)$ is of form $P\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$, where P is a polynomial, hence that f is of class $\mathcal{C}^n$, for any integer n, with a zero derivative in 0.

f is therefore of class $\mathcal{C}^\infty$ and this Taylor-series is zero. If f could be developed in power series, its development (by unicity) would be the zero function around 0, whereas f is not zero around 0.
The the previous counter-example the Taylor-series converges, but not function f. We also can find cases where the Taylor-series has a zero radius of convergence.

d) Sufficient existance condition of the development as a power series of a function around 0
Theorem
Let f be a function from $\mathbb{R}$ to $\mathbb{R}$, of class $\mathcal{C}^\infty$ on an interval of the type $]-a,a[$.
If there exists $M \in \mathbb{R}$ such that: $\forall n \in \mathbb{N},\ \forall x \in]-a,a[,\ \left|f^{(n)}(x)\right| \leqslant M$, then the power series $\sum\limits_{n\geq 0} \frac{f^{(n)}(0)}{n!}x^n$ converges simply on $]-a,a[$ and its sum is equal to f.

2) Development as a power series of usual functions
a) Exponential function and associated functions
$e^x = \sum\limits_{n=0}^{+\infty} \frac{x^n}{n!}$, the radius of convergence is infinite, by d'Alembert's rule.
The complex exponential has been directly defined by its development as a power series and allows us to obtain the developments of sinus and of cosinus:
$\cos x = \mathrm{Re}\left(e^{ix}\right),\ \sin x = \mathrm{Im}\left(e^{ix}\right)$, hence:

$$\sin(x) = \sum_{n=0}^{+\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{+\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

From the development of the exponential, we also infer those of ch and sh:

$$\mathrm{sh}(x) = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\mathrm{ch}(x) = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}$$

All radiuses of convergence are infinite.
The five previous series have an infinite radius of convergence, which implies the normal convergence of the series on any segment, but not the normal convergence on $\mathbb{R}$.
In fact there is never a normal convergence on $\mathbb{R}$ for a power series, except if it's a polynomial.

b) Function $x \to (1+x)^a, a \in \mathbb{R}$
The function f, defined on $]-1,+\infty[$ by $f(x) = (1+x)^a$, $a \in \mathbb{R}$, can be developed

as a power series around 0 and:
$\forall\ x \in]-1,1[, (1+x)^a = 1 + \sum_{n=1}^{+\infty} \frac{a(a-1)\ldots.(a-n+1)}{n!} x^n$.
The radius of convergence is equal to 1, except if we cannot apply the d'Alembert's rule because one of the a_n coefficients is zero, which implies that a is a natural integer. Function f is therefore a polynomial and it has a development as a power series of infinite radius.

Important special cases
i) If a is a natural integer, the coefficients are zero after a certian rank, the series is therefore reduced to a polynomial, which corresponds to the binomial formula, and this case the radius is infinite.

ii) $a = -1$, we find the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n, \forall x \in]-1,1[$$

iii) We infer, replacing x by $-x$:

$$\frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n,\ \forall x \in]-1,1[$$

iv) $a = \frac{1}{2}$

$$\sqrt{1+x} = 1 + \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1.3\ldots(2n-3)}{2.4\ldots 2n} x^n,\ \forall x \in]-1,1[$$

v) Integrating the first two series, we can find the developments as power series of $\ln(1-x)$ and $\ln(1+x)$:

$$\ln(1-x) = -\sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1},\ \forall x \in]-1,1[.$$

$$\ln(1+x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1},\ \forall x \in]-1,1[.$$

These two series are also of radius 1.

v) $a = -\frac{1}{2}$
$a_n = \frac{-\frac{1}{2}\left(-1-\frac{1}{2}\right)\ldots\left(-n+1-\frac{1}{2}\right)}{n!} = (-1)^n \frac{1.3\ldots(2n-3)}{2.4\ldots 2n}$ and

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{+\infty} (-1)^n \frac{1.3\ldots(2n-3)}{2.4\ldots 2n} x^n,\ \forall x \in]-1,1[.$$

vi) Replacing x by x^2 in the developments of $\frac{1}{1+x}$, $\frac{1}{1-x}$ and $\frac{1}{\sqrt{1-x}}$, we obtain the developments of: $\frac{1}{1+x^2}$, $\frac{1}{1-x^2}$ et $\frac{1}{\sqrt{1-x^2}}$, and by integration we obtain those of: Arctgx, argth$x$, $\arcsin x$.

3) Development as a power series in a point other than 0

We just have to replacer the Taylor-series in 0 (also called Mac-Laurin) by the Taylor-series at point a.

Let f be a function defined from $\mathbb{R}$ to $\mathbb{R}$. We say that f can be developed as a power series around point a if there exists an interval I of center a such that:

$$\forall x \in I,\ f(x) = \sum_{n=0}^{+\infty} a_n (x-a)^n.$$

Function f is then $\mathcal{C}^\infty$ on I, and $a_n = \frac{f^{(n)}(a)}{n!}$.

Theorem

If function f can be developed as a power series around 0, it is equal to the sum of its Taylor-series in 0 on an interval $]-R, R[$.

Then function f is equal to the sum of its Taylor-series in a, for any a element of $]-R, R[$, meaning:

$$\forall x \in]-R, R[\ /\ |a| + |x-a| < R,\ f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Exercises

1) Find the radius of convergence of the following power series:

$a)$ $\sum\limits_{n\geq 0} z^n$.

b) $\sum\limits_{n\geq 0} \frac{z^n}{n!}$.

c) $\sum\limits_{n\geq 0} n^n z^n$.

d) $\sum_{n\geq 1} \ln(n) z^n$

e) $\sum\limits_{n\geq 1} a_n z^n$ where a_n is the n-th decimal of π.

2) Find the radius of convergence of the power series:
$\sum_{n=1}^{\infty} \frac{n^n}{3^{n^2}} x^n$

3) Find the radius of convergence of the power series:

$$\sum_{n\geq 0} a_n x^n \text{ avec } \begin{cases} a_{2n} = 2^n \\ a_{2n+1} = \cos(2n) \end{cases}$$

4) Find the radius of convergence R of the power series $\sum_{n=1}^{+\infty} \frac{ch(n)}{n} x^n$.
For $x \in \mathbb{R}$ such that $|x| < R$, find the sum of the series.

5) Find the radius of convergence R of the power series:
$\sum_{n=0}^{+\infty} \frac{n^2+4n-1}{n+2} x^n$.
For any real number x such that $|x| < R$, calculate the sum of the series.

6) Find the radius of convergence and calculate the sum of the power series:

$\sum_{n=0}^{+\infty} \frac{x^n}{2n+1}$.

7) Find the radius of convergence and calculate the sum of the power series:
$\sum_{n\geq 1} \frac{(-1)^n x^{2n+1}}{3^{n-1}(2n+1)}$.

8) We consider the real power series:
$\sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n-1)}$.
a) Find the radius and the domain of convergence of the power series.

b) Calculate the sum of the series and determine the domain of continuity of the sum.

c) Infer the sum of the series: $\sum_{n\geq 1} \frac{(-1)^n}{4n^2-1}$

9) We denote by $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ two real sequences such that: $\begin{cases} u_{n+1} = u_n + 2v_n \\ v_{n+1} = u_n + v_n \end{cases}$.
Find the radius of convergence of the power series with general terms $\frac{u_n x^n}{n!}$ and $\frac{v_n x^n}{n!}$.

10) We denote by f and g the sums of two power series of same radius 1.
We set, for any complex number z with an absolute value strictly lower than 1:
$f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n, \ g(z) = \sum\limits_{n=0}^{+\infty} b_n z^n$.
Justify the term by term integration of the product series at point $\frac{e^{i\theta}}{2}$, $\theta \in [0, 2\pi]$.
Infer: $\int\limits_0^{2\pi} f\left(\frac{e^{i\theta}}{2}\right) g\left(\frac{e^{i\theta}}{2}\right) d\theta$.

11) We denote by $(a_n)_{n\in\mathbb{N}}$ a convergent sequence of complex numbers.
Show that the power series $\sum_{n\geq 0} a_n z^n$ has a radius of convergence greater or equal to 1.
What more can we say if we assume that the limit is non zero ?

12) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers of limit a when n goes to infinity.
a) What is the radius of convergence of the power series $\sum\limits_{n\geq 0} \frac{a_n x^n}{n!}$?
b) We assume that $a = 0$.
Let f be the sum of the series, show that: $\lim\limits_{x\to+\infty} e^{-x} f(x) = 0$.
c) Generalise this result to the case of an arbitrary limit a.

13) Let $\sum_{n\geq 0} a_n z^n$ be a power series, we assume that the power series $\sum_{n\geq 0} a_{2n} z^{2n}$ and $\sum_{n\geq 0} a_{2n+1} z^{2n+1}$ are of radius of convergence R_1 and R_2.
What is the radius of convergence of the power series $\sum_{n\geq 0} a_n z^n$?

14) Find the radius of convergence and calculate the sum of the power series:

$\sum_{n\geq 0} \frac{x^{3n}}{(3n)!}$

15) We denote by $\sum_{n\geq 0} a_n z^n$ a power series of radius 1.
We set: $s_n = \sum_{k=0}^{n} a_k$ and $t_n = \frac{s_n}{n+1}$.
a) Find the radius of convergence R of the power series:
$\sum_{n\geq 0} s_n z^n$ and $\sum_{n\geq 0} t_n z^n$.

b) For any complex number z with an absolute value strictly lower than 1, we set $f(z) = \sum_{n=0}^{+\infty} a_n z^n$.
Find dependind on $f(z)$ the sum of the power series $\sum_{n\geq 0} s_n z^n$.

16) Let $\sum_{n\geq 0} a_n z^n$ be a power series of radius of convergence R.
Find the radius of convergence of the power series:
a) $\sum_{n\geq 1} ln(n) a_n z^n$.

b) $\sum_{n\geq 0} e^n a_n z^n$

c) $\sum_{n\geq 0} P(n) a_n z^n$, where P is a polynomial.